Restricted Coloring Problems and Forbidden Induced Subgraphs

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Abstract

An *acyclic coloring* of a graph is a proper coloring such that any two color classes induce a forest. A *star coloring* of a graph is an acyclic coloring with the further restriction that the forest induced by any two color classes is a disjoint collection of stars. We consider the behavior of these problems when restricted to certain classes of graphs. In particular, we give characterizations of the classes of graphs for which two or more of these restricted coloring problems are equivalent, in that they share the same set of solutions. Surprisingly, our characterizations of these classes in terms of forbidden induced subgraphs equate them with classes that are well-studied in the literature. We extend this framework to encompass other restricted coloring problems, both known and new, and outline a method for obtaining results similar to those given here.

We also explore the algorithmic implications of these results in terms of finding optimal acyclic and star colorings on certain classes of graphs. We show that optimal acyclic colorings of certain subclasses of even-hole-free graphs can be found in polynomial time, and that optimal acyclic and star colorings of trivially perfect graphs can be found in linear time.

1 Introduction

A *proper coloring* (or *coloring*) of a graph $G = (V, E)$ is a mapping $\phi : V \to \mathbb{N}^+$ such that if $ab \in E$ then $\phi(a) \neq \phi(b)$. An *acyclic coloring* of a graph is a proper coloring such that the subgraph induced by the union of any two color classes is a forest. A *star coloring* of a graph is a coloring such that the subgraph induced by the union of any two color classes is a disjoint collection of stars. Let $\Phi(G)$ denote the set of all colorings of a graph G. Similarly, let $\Phi_a(G)$ and $\Phi_s(G)$ denote the sets of acyclic and star colorings of G, respectively. The following statement follows from the observation that a disjoint collection of stars constitutes a forest.

Proposition 1.1. *For every graph G, $\Phi_s(G) \subseteq \Phi_a(G) \subseteq \Phi(G)$.* □

In the first part of this paper, we discuss the properties of acyclic and star colorings when restricted to particular classes of graphs. Consider the following recent result.

Theorem 1.2 (Gebremedhin et. al. [8])**.** *If G is a chordal graph, then $\Phi(G) = \Phi_a(G)$.*

In Section 2, we supplement Theorem 1.2 with a sufficient condition in order to find a characterization of the graphs for which $\Phi(G) = \Phi_a(G)$; it is shown that they are exactly the even-hole-free graphs. Moreover, we show that the graphs for which $\Phi_a(G) = \Phi_s(G)$ are exactly the cographs, and the graphs for which $\Phi(G) = \Phi_s(G)$ are exactly the trivially perfect (or quasi-threshold) graphs.

All of the coloring problems discussed so far are **NP**-complete on general graphs, which motivates our search for tractable restricted classes. In Section 3 we shift our focus to the algorithmic complexity of finding optimal restricted colorings on special classes of graphs, beginning with a broad summary of known algorithms and hardness results for acyclic and star coloring. We focus on these particular coloring problems in part because of their applications to the optimal evaluation of sparse Hessian matrices, where the graph corresponds to the sparsity structure of matrix, which is symmetric. The star and acyclic coloring problems correspond to direct and indirect schemes for recovery of the Hessian, respectively. See [10] for a survey of coloring problems as they relate to sparse derivative matrices.

The authors of [8] discuss the case when the Hessian matrix is known in advance to have banded structure. It is shown that the resulting class of graphs is properly contained within the class of chordal graphs. Consequently, Theorem 1.2 implies that any solution to the classical coloring problem is also a solution to the acyclic coloring problem. Because chordal graphs can be properly colored in $O(n+m)$ [12, Chapter 4.7], the same is true for acyclic coloring. We explore this concept in more depth in Section 3, where we give similar, more general results that follow from the characterizations given in Section 2.

In Section 4 we outline the extension of and possible generalizations of these ideas to coloring problems with other restrictions, both known and new, which form a poset with respect to inclusion on their respective sets Φ of all valid colorings. Each of the classes mentioned above has a nice characterization in terms of forbidden induced subgraphs; we discuss the possible use of this as a framework for a more general description of the classes of graphs for which certain relationships occur between different restricted coloring problems.

2 When are two restricted coloring problems equivalent?

A graph is *even-hole-free* if it contains no induced cycle with an even number of vertices. Even-hole-free graphs can be recognized in polynomial time [3], and are a superclass of the chordal graphs.

The following theorem appears as an unproven observation in [13]. We include a proof here for the sake of completeness.

Theorem 2.1. *A graph G is even-hole-free if and only if every coloring of G is also an acyclic coloring.*

Proof. $(\Rightarrow)$: Assume that G is even-hole-free. We will show that any distance-1 coloring of G is also an acyclic coloring. Any odd cycle will certainly require at least three colors, so we restrict our attention to even cycles. Let C be some even cycle in G. Because G is even-hole-free, C must have at least one chord ab which we can view as the intersection of two smaller cycles C_1, C_2 formed by vertices from C. Since C is an even cycle, either $|C_1|$ and $|C_2|$ are both even or both odd. If both $|C_1|$ and $|C_2|$ are odd, then they both require three distinct colors. If both $|C_1|$ and $|C_2|$ are even, then treat them each recursively. Proceeding by induction over the size of C, we conclude that C must use at least three distinct colors. Thus any distance-1 coloring of G is also an acyclic coloring, and we are done with this direction.

$(\Leftarrow)$: Let G have the property that every distance-1 coloring of G is also an acyclic coloring, and assume for the sake of contradiction that G contains an even induced hole. Then there exists at least one coloring of G that uses only two colors for the vertices in C. As an example, we may take the coloring that assigns the colors $1, 2$ alternating around the vertices in C, and assigned every vertex not in C its own distinct color. The result will be a distance-1 coloring that is not an acyclic coloring, which is a contradiction. Thus G must be even-hole-free, which completes the proof. □

Corollary 2.2. *If G is an even-hole-free graph, then $\chi(G) = \chi_a(G)$.*

We now prove a result for star coloring that is analogous to Theorem 2.1. A graph G is *trivially perfect* if for every induced subgraph G' of G the number of maximal cliques in G' is equal to the size of the largest independent set in G'. Trivially perfect graphs were first studied by Wolk [23], who encountered them while searching for a characterization of comparability graphs. Other names for this class include *quasi-threshold* graphs [2], and, as illustrated in Theorem 2.3(iii), P_4-free chordal graphs [18, Ch. 7.9].

Theorem 2.3 (Golumbic [11], Yan et. al. [24])**.** *Let G be a graph. Then the following conditions are equivalent:*

(i) *G is trivially perfect;*

(ii) *G contains no induced C_4 or P_4;*

(iii) *G is both chordal and a cograph.*

Theorem 2.4. *A graph G is trivially perfect if and only if every coloring of G is also a star coloring.*

Proof. $(\Leftarrow)$: Let G be a graph such that $\Phi_s(G) = \Phi(G)$. If G contains an induced subgraph $G' \in \{C_4, P_4\}$, then there must exist some coloring $\phi \in \Phi(G)$ that colors G' using only two colors, which means that $\phi \notin \Phi_s(G)$, a contradiction. It follows from Theorem 2.3 that G is trivially perfect.

$(\Rightarrow)$: Suppose G is trivially perfect, and assume for the sake of contradiction that there exists some coloring $\phi \in \Phi(G)$ such that $\phi \notin \Phi_s(G)$. Then for some P_4abcd in G we must have $\phi(a) = \phi(c) = i$ and $\phi(b) = \phi(d) = j$, where $i \neq j$. Because G has no induced P_4, at least one of the edges ad, ac, or bd must be present in G. If $ad \in E$, then $\{a, b, c, d\}$ induces a C_4 unless ac or bd is also present, where the presence of either one contradicts the 2-colorability of the P_4. Therefore we have $\Phi_s(G) = \Phi(G)$, which is the desired result. □

Corollary 2.5. *If G is a trivially perfect graph, then $\chi(G) = \chi_a(G) = \chi_s(G)$.*

Corollary 2.6. *A graph is trivially perfect if and only if it is both even-hole-free and a cograph.*

A graph is a *cograph* if it has no induced P_4. Cographs can be recognized in linear time [6].

Theorem 2.7. *Let $G = (V, E)$ be a graph. Then G is a cograph if and only if every acyclic coloring of G is also star coloring.*

Proof. $(\Rightarrow)$: Assume G is a cograph, and let ϕ be an acyclic coloring of G. Furthermore, let $P = abcd$ be a (not necessarily induced) P_4 in G; we will show that ϕ assigns at least three different colors to the vertices in P. Because G is a cograph (and thus contains no induced P_4, we know that at least one of the following is true: (1) $ad \in E$, (2) $ac \in E$, (3) $bd \in E$. If $ad \in E$, then $abcd$ constitutes a C_4 in G, and because ϕ is an acyclic coloring, ϕ must assign at least three colors to the vertices in P. Otherwise, assume without loss of generality that $ac \in E$ (the case where $bd \in E$ holds symmetrically), and observe that abc is a triangle in G. This implies that ϕ assigns at least three colors to P, which completes this direction.

$(\Leftarrow)$: Let G have the property that every acyclic coloring ϕ of G is also star coloring, and assume for the sake of contradiction that there exist vertices $w, x, y, z \in V$ that induce a P_4 in G (in that order). It follows that there exists some acyclic coloring ϕ such that $\phi(w) = \phi(y)$ and $\phi(x) = \phi(z)$. Such a coloring makes $wxyz$ a bichromatic P_4, which is a contradiction. Thus G cannot contain an induced P_4, and we may conclude that G is a cograph, as desired. □

Corollary 2.8. *If G is a cograph, then $\chi_a(G) = \chi_s(G)$.*

Corollary 2.9. *If ϕ is a proper coloring of a cograph G, then either ϕ causes a bichromatic cycle in G or ϕ is also a star coloring.*

3 Algorithms and complexity on restricted classes

Let us now turn our attention towards algorithms for finding optimal acyclic and star colorings on special classes of graphs. As we saw in the case of chordal graphs, algorithms for one coloring problem can be readily adapted for solving a different coloring problem when the input graph belongs to a particular class. In this section, we state some results that follow in the same vein from Theorems 2.1, 2.4, and 2.7 after reviewing what is known about finding optimal acyclic and star colorings on special classes of graphs. We also present some open problems related to this work. Before doing so, however, we briefly discuss the nature of algorithms designed to work on restricted input domains.

Promise vs. robust algorithms. We consider two different types of algorithms for solving problems on particular classes of graphs. A *promise* algorithm assumes that the input is in the restricted class, and may have unknown or undefined behavior when this is not the case. It is reasonable to consider promise algorithms exclusively in the case of chordal graphs, as membership in this class can be decided in linear time. Thus when we are given a graph to acyclically color, we can first check and see whether the graph is chordal, and proceed with our promise algorithm if this is the case. If the graph is not chordal, we may either try again with a different class or simply proceed with an heuristic or approximation algorithm.

Informally, an algorithm that solves a problem on a restricted domain is called *robust* ([20],[22, Ch. 14]) if it doesn't malfunction when the input doesn't fall in the domain of interest. More precisely, a robust algorithm for a problem π on a class $\mathcal{C}$ must behave in the following way. If the input is in the class $\mathcal{C}$, then the algorithm produces the correct output. Otherwise, the algorithm must either produce the correct output or return a certificate that the input is not in class $\mathcal{C}$.

Thus for chordal graphs, and any other class that can be recognized efficiently, a promise algorithm can be made robust by first running a recognition algorithm. We must be careful in doing so, however, because there are many classes for which no linear time recognition algorithm is known. The resulting robust algorithm will run in time proportional to the sum of the running times of the recognition and promise algorithms.

Finding optimal acyclic colorings. Coleman and Cai [4] discovered the notion of acyclic coloring as a formulation of a problem related to the estimation of sparse Hessian matrices via substitution methods. Unaware of the work in the graph theory literature, they called it "cyclic" coloring. They demonstrated that this problem is **NP**-complete even for bipartite graphs. Many of the known algorithms for acyclic coloring are for graphs with bounded maximum degree [21, 7]. In particular, graphs with maximum degree ≤ 3 can be colored with 4 or fewer colors in linear time, and graphs with maximum degree ≤ 5 can be colored with 9 or fewer colors in linear time. Note that these algorithms do not necessarily find optimal colorings.

Suppose we are given a graph G and asked to find an optimal acyclic coloring. We now discuss ways in which Theorem 2.1 might be of some use in finding such a coloring. It is not currently known whether even-hole-free graphs can be colored optimally in polynomial time. However, if G is an arbitrary graph, we can determine in polynomial time whether G is even-cycle-free [3] and, if so, we can use one of the many heuristics and approximation algorithms for proper coloring general graphs. It follows from Theorem 2.1 that approximation bounds and other properties of such algorithms would also hold when they are applied to acyclic coloring on even-hole-free graphs. In some cases, we may know beforehand that the graphs given to us will have special structure. We may, for instance, be working with an application for which it is known that the matrices involved will always correspond to even-hole-free graphs. If no such structure is known, however, we may be tempted to attempt to recognize arbitrary graphs as even-hole-free. However, given that it is not known whether this can be done in a way that is practical, it is unlikely that such a strategy will be more economical than simply running general coloring heuristics and approximation algorithms on general graphs, subsequently checking the colorings they produce for bichromatic cycles, which can be done in polynomial time. The bottleneck here is currently recognizing even-hole-free graphs; the best known algorithm runs in $O(n^{15})$ time [3]. Ideally, we would want the stronger result, which would be an algorithm for coloring (and thus acyclicly coloring) the entire class of even-hole-free graphs. Assume for the moment that we have access to a fast ($o(n^{15})$, say) algorithm for coloring even-hole-free graphs, and we wish to use this algorithm in a context where we are given a general graph and asked to find an optimal acyclic coloring. If we proceed by first trying to recognize whether the graph is even-hole-free (using the $O(n^{15})$ algorithm), then the speed of our fast coloring algorithm will be irrelevant, as the cost of the recognition step will dominate.

However, we may avoid being constrained by the complexity of the recognition problem, as it is not necessary that we use a promise algorithm. What is really desired is a robust algorithm such as the following.

Problem 1. *Devise a polynomial-time algorithm that when given a graph G returns either an optimal acyclic coloring or a certificate that G isn't even-hole-free.*

Note that the algorithm proposed in Problem 1 does not act as a recognition algorithm, as it may also return an optimal acyclic coloring for graphs that aren't even-hole-free.

Another possibility is to consider subclasses of even-hole-free graphs for which coloring can be solved efficiently. For example, Theorem 1.2 gave us a linear-time algorithm for acyclic coloring on chordal graphs. Another such subclass is the β-perfect graphs, which can also be colored efficiently. Whether these graphs can be recognized efficiently, however, is an open problem (though it is known to be in co-**NP** [2]).

Similarly, the (even-hole,diamond)-free graphs are a subclass of the β-perfect graphs that can be colored (and thus acyclicly colored) in polynomial time. It was also shown in [16] that the (even-hole,diamond)-free graphs are a subclass of the β-perfect graphs, which in turn are a subclass of the even-hole-free graphs.

Corollary 3.1. *There is a polynomial time robust algorithm for finding an optimal acyclic coloring of (even-hole,diamond)-free graphs.*

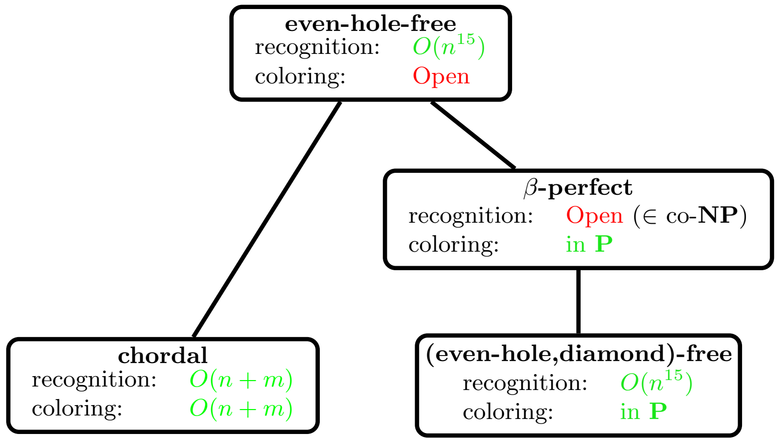

Figure 1: Some classes of graphs that are even-hole-free.

Finding optimal star colorings. We discussed algorithms for star coloring cographs and trivially perfect graphs at the beginning of this section. As the boundary between tractability and intractability is of interest for this problem, we note here some **NP**-completeness results related to optimal star coloring of graphs. Coleman and Moré [5] showed that the decision variant of this problem is **NP**-complete even for bipartite graphs. The following results were shown in [1].

(i) It is **NP**-complete to determine whether $\chi(G) = \chi_s(G)$, even is G is a planar graph with $\chi(G) = 3$.
(ii) It is **NP**-complete to determine whether a graph can be star colored with 3 colors, even for planar bipartite graphs.
(iii) For $2 \le t \le k$ and $k > 2$, given a graph G with $\chi(G) = t$, it is **NP**-complete to decide if $\chi_s(G) \le k$.

Given that we can find an acyclic coloring of a chordal graph in linear time, we might wonder whether the same is true for star coloring. A graph G is a *split graph* if its vertices can be partitioned into sets V_1 and V_2 such that V_1 induces a clique and V_2 induces an independent set. The split graphs are a well-known subclass of the chordal graphs.

Problem 2. *Find a polynomial time algorithm or show that it is* **NP***-complete to determine whether* $\chi(G) = \chi_s(G)$ *for a split graph* G.

Finding optimal acyclic and star colorings. Trivially perfect graphs can be recognized and optimally colored in linear time [24]. The following result implies that an optimal star coloring of a quasi-threshold graph can also be found in linear time. Note that as a subclass of the interval graphs, these can be colored easily in linear time. The following result follows from Theorem 2.3(iiii), which states that every trivially perfect graph is also chordal.

Corollary 3.2. *There is a linear-time robust algorithm that solves the acyclic and star coloring problems for trivially perfect graphs.*

As the trivially perfect graphs are a subclass of the cographs, Corollary 3.2 also follows from the following theorem.

Theorem 3.3 ([17])**.** *There is a linear time robust algorithm for finding acyclic and star colorings of cographs.*

4 Generalization to other restricted coloring problems

We consider here the generalization of this framework to other restricted coloring problems. The main idea behind our approach hinges on viewing restricted colorings in terms of forbidden subgraphs in the graphs induced by any two color classes. We give the following propositions (without proof), which indicate some implications of the suggested framework.

Proposition 4.1. *A coloring ϕ of a graph G is proper if and only if the subgraph G' induced by any two colors is $\boldsymbol{C_{2k+1}}$-free (G' has no induced odd cycles).* □

Proposition 4.2. *A coloring ϕ of a graph G is an acyclic coloring if and only if the subgraph induced by any two colors is $\boldsymbol{C_k}$-free.* □

Proposition 4.3. *A coloring ϕ of a graph G is a star coloring if and only if the subgraph induced by any two colors is ($\boldsymbol{P_4}$,$\boldsymbol{C_k}$)-free.* □

There is another coloring variant which arises out of both graph theoretical and applied settings (again the evaluation of sparse derivative matrices). A *distance-2 coloring* is a proper coloring with the additional restriction that no vertex shares its color with a distance-2 neighbor. We give the following without proof, where part (ii) will be the most relevant for our purposes.

Proposition 4.4. *Let ϕ be a coloring of a graph G. The following conditions are equivalent:*

(i) *ϕ is a distance-2 coloring;*
(ii) *The subgraph induced by any two colors is ($\boldsymbol{P_3}$,$\boldsymbol{C_3}$)-free;*
(iii) *The subgraph induced by any two colors is an induced matching;*
(iv) *The subgraph induced by any 3 vertices is disconnected.* □

A coloring ϕ is a *linear coloring* [25] if the subgraph induced by any two color classes is a disjoint collection of paths. A more appropriate name for our framework is *path coloring*.

Proposition 4.5. *A coloring ϕ is a linear (path) coloring if and only if the subgraph induced by any two colors is ($\boldsymbol{K_{1,3}}$,$\boldsymbol{C_k}$)-free.* □

A graph G is a *caterpillar* if it has a dominating path.

Definition 1 (caterpillar coloring)**.** *A coloring ϕ of a graph G is a* caterpillar coloring *if the subgraph induced by any two color classes is a disjoint collection of caterpillars.*

Proposition 4.6. *A coloring ϕ is a caterpillar coloring if and only if the subgraph induced by any two colors is ($\boldsymbol{T_2}$,$\boldsymbol{C_k}$)-free.* □

Proposition 4.7. *A graph G is claw-free if and only if every proper coloring either 2-colors a cycle or the subgraph induced by any two colors is a disjoint collection of paths.* □

Proposition 4.8. *A graph G is (even-hole,claw)-free if and only if every proper coloring of G is also a path coloring.* □

Thus far, we have considered only colorings for which the connected components of every 2-colored induced subgraph are trees. We may also consider other types of bipartite graphs. A graph G is a *chordal bipartite graph* if G is bipartite and every cycle of length greater than 4 has a chord. A coloring ϕ of a graph G is a *chordal coloring* if the subgraph induced by any two color classes is a disjoint collection of chordal bipartite graphs.

Proposition 4.9. *A coloring ϕ is a chordal coloring if and only if the subgraph induced by any two colors is ($\{\boldsymbol{C_k} : k \neq 4\}$)-free.* □

We might also consider other classes of bipartite graphs. For example, we might require that every connected component in the subgraph induced by any two color classes is a chain graph ($2K_2$-free).

As a general suggestion, we propose the following framework in which to classify and study restricted coloring problems. Let Π be a (possibly infinite?) collection of coloring problems. We define a poset $P_\Pi = \langle \Pi, \leq_\chi \rangle$, where for any $\pi_1, \pi_2 \in \Pi$ we have $\pi_1 \leq_\chi \pi_2$ if and only if $\Phi_{\pi_1} \subseteq \Phi_{\pi_2}$. We may begin by considering the resulting poset on all classes of graphs, and investigate how it relates to the corresponding poset when restricted to particular classes of graphs. For example, we have seen that if G is even-hole-free, then the classical coloring and acyclic coloring problems collapse. We may view this as a contraction of their respective elements in P_Π.

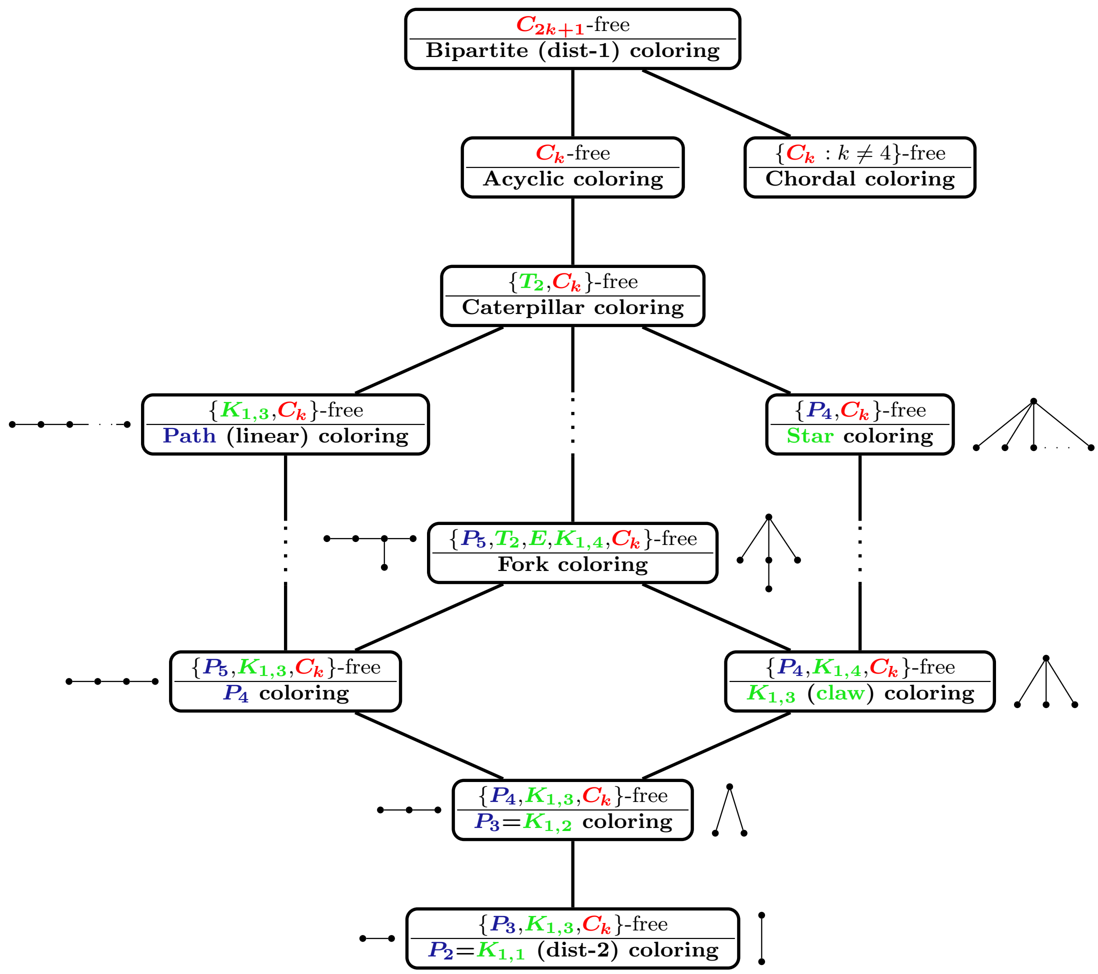

Figure 2: A collection of restricted coloring problems ordered by containment. Shown here is the Hasse diagram of the lattice ordered set $P_\Pi = \langle \Pi, \leq_\chi \rangle$.

5 Concluding remarks

All the coloring problems discussed in this paper have multiple equivalent formulations. Acyclic and star colorings can be defined by the requirement that every cycle or P_4 uses at least three colors, respectively. We have seen that these problems can also be defined in terms of the structure of the subgraphs induced by every pair of color classes, which is a forest in the case of acyclic coloring and a disjoint collection of stars in the case of star coloring. The latter approach has proven useful in designing practical heuristics for large instances [9]. Additionally, star coloring has an equivalent formulation in terms of *in-colorings* of oriented graphs [19, 1, 15].

In Section 4, we proposed a new framework which is especially useful for studying the behavior of these problems on restricted classes of graphs. In particular, we suggest that coloring problems can be considered in terms of forbidden induced subgraphs in the graphs induced by pairs of color classes. This framework has be shown to provide a connections between acyclic and star coloring, as well as other problems mentioned in the literature, such as path (or linear) and distance-k coloring. From this perspective, it is possible to leverage what is known about graphs defined by forbidden induced subgraphs to identify graph classes which have particular behavior with respect to these problems. In many cases, as shown here, this also allows us to exploit the properties of such graph classes in order to design efficient algorithms. It is our hope that advances in this direction will be useful for applications that arise in scientific computing.

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